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IMPERFECTIONS, A MAIN CONTRIBUTOR TO SCATTER IN EXPERIMENTAL VALUES OF BUCKLING LOAD

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Abstract—This paper develops a novel approach for testing cylindrical shells. A method of buckle restriction by means of an internal mandrel is used. This mandrel is located at a distance equivalent to the thickness of the shell from the surface of the test specimen. By means of this device it is possible to elastically buckle the shell until the shell is completely filled with buckles. It is shown that the variation of the rate of change in number of buckles as a function of the applied load follows a Gaussian distribution. The mode of the Gaussian curve is located at a load level which agrees very closely with the critical buckling load computed from the classic formula.

It is considered that the evidence presented is substantial support for the small displacement theory and a strong indication that buckling takes place first at the weakest section of the shell and the amount of buckling is a function of the area of weakness.

1. INTRODUCTION

TODAY much data exists with regard to the buckling of plates and shells. The chief conclusions of both the theoretical and experimental investigations are outlined in detail in several papers [1, 2]. Unfortunately, the results of experiments with circular cylindrical shells are not at the present time consistent with theoretical predictions. The values of buckling loads determined by tests differ appreciably from the computed values, no matter what theory we use.

Attempts to explain these differences in behavior have been made by various researchers. Donnell [3] attributed early failure to initial deviations from the geometrically exact shape. Flügge [4] calculated the stresses caused in the shell by the restraint to expansion provided by the testing machine. Both of these workers were able to demonstrate that reductions in failing load would occur from these causes but they did not account for the large deviations in observed buckle shape from those predicted by theory.

In part, the discrepancy may be due to the fact that the buckle pattern observed is not of the type normally used in theoretical considerations. It is most unusual in practice for the buckle pattern to develop uniformly over the whole tube. Rather the characteristic diamonds are formed over parts of the tube. A wide field of theoretical exploration would be opened up if an attempt were made to cover localized bands of diamond buckles. To some extent this has already been attempted by Hoff [2].

The purpose of the study reported herein was to check whether the influence of initial irregularities overshadows all consideration of elaborate waveforms, as is suggested by Cox [5].

The fact that initial imperfections have a pronounced effect on the value of the first buckling load and the point at which it occurs is supported by a result published by the authors in a previous paper [6]. In the study reported in this paper an electroformed nickel cylinder was tested under compression and the buckle load history is shown in curve 1 of Fig. 1. This buckling occurred in the lower 20% of the shell. As a result of this

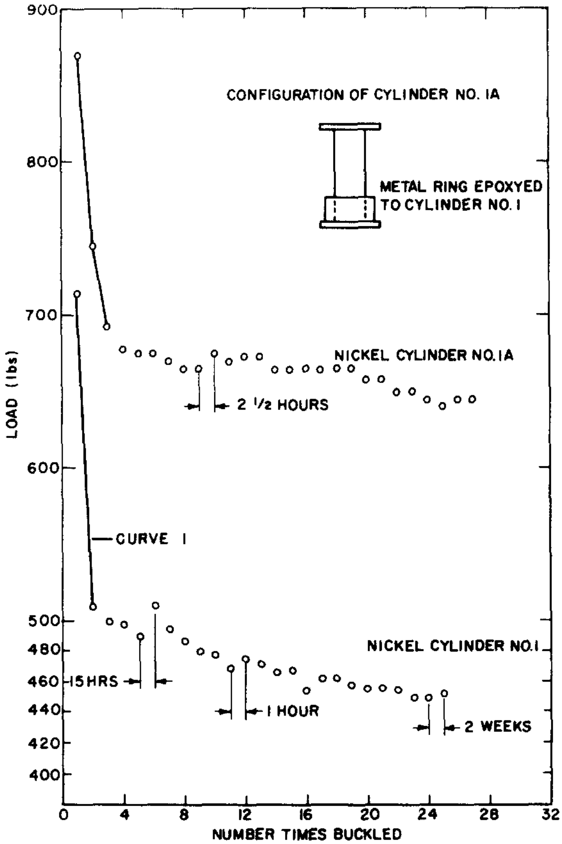

FIG. 1. Repeated buckling behavior of a cylindrical shell.

we were able to make a closely fitting heavy collar which was glued around the shell in such a manner as to reduce the effective length of the shell. The compression test was carried out again and in this case failure occurred at a higher load than had previously been experienced. Subsequent measurement showed that the buckles occurred at the most irregular positions on the cylinder. Thus, our experiment demonstrated among other things that the load and position at which buckling occurs is a function of geometric imperfection.

The testing of cylindrical shells is a subject fraught with difficulties. Even if a family of shells with prescribed small local variations in form and property could be made a large number of tests would be required. The test results achieved on the family would be influenced by many factors other than those to be investigated. For example, the machine used, the individual setup, the nature of the end fittings and so on, would influence the results. It is clear that variation due to these causes must be eliminated in any test series. Hence, some radically new approach to testing of shells was necessary if the question posed was to be answered experimentally.

We reconsider the question of a family of specimens in the following manner: n cylindrical shells manufactured by the same processes to identical specification should have a random variation in irregularity unless some bias be introduced by the inspection process. These n specimens could be placed together in a continuous line and if we were to examine them in this form the random variation would still occur. If now they were joined together they would produce a longer shell which would be characterized by the same normal distribution of irregularity over its length as had previously been found in the individual members which had been bonded together to form the single specimen. Thus, the single specimen has basically the character of a set.

If this single specimen can be tested in an appropriate manner then it should be possible from the one test specimen to obtain the characteristics of a family of cylinders. In previous researches the authors have demonstrated [7] that when the depth to which a buckle is permitted to develop is restricted to the thickness of the shell then the buckle process has no deleterious effect on subsequent buckle behavior of the shell. Moreover, if a restriction of this kind is made by means of some interior object lying close to the surface of the shell it is possible, with increasing load, to produce a family of buckles which will cover the entire surface of the shell. Figure 2 shows a well-developed buckle pattern on a cylindrical shell tested in this manner.

It would seem reasonable under these circumstances, that for an initially stress free cylinder the number of buckles formed would be a measure of the geometric and material imperfection. Additionally, the rate of change in number of buckles versus the applied load would follow a normal distribution and the average of the load at which buckles first were realized and the load at which the shell was completely filled with buckles would correspond closely to the buckling load of a perfect cylinder of the nominal thickness. If the contention of Cox is correct, this load should have the value $0{\cdot}6(Et/r) \times 2\pi rt$ approximately.

We should anticipate that the better the manufacturing techniques used for the shells the more leptokurtic would the distribution curve be and, of course, the poorer the technique the more platykurtic. (See Fig. 3). It is not unlikely, too, that with some

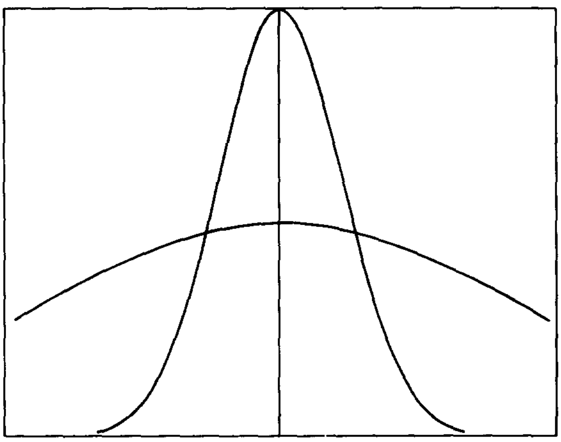

FIG. 3. Kurtosis of distribution curve.

methods of manufacture and with some materials a measure of skewness will be apparent. This would certainly be the case, for example, if in the cylinder under test the stress–strain curve for the material is not Hookean over the complete range of stresses which result from the loading range required.

2. NATURE OF THE TEST SPECIMENS

The first cylindrical shells used in this study were manufactured by electroforming (Fig. 4). A thin coat of nickel was deposited on an accurate aluminum mandrel. The shells were separated from the mandrels by rapidly cooling the mandrels with liquid nitrogen. The finished shells were 0·004 in wall thickness, 2·906 in. diameter and 8 in. length. The nickel had a Young's modulus of approximately 24×10^6 lb in^{-2}, as determined in a separate test.

Thus, the classical buckling load of the specimens was computed from the formula

$$P_{crit} = 0{\cdot}6 \frac{Et}{r} \times 2\pi rt = 1455.$$

3. METHOD OF TEST

The specimen was mounted in the special test fixture, shown in Fig. 5. It is clear from this drawing that the fixture differs from the normal 'encastré' type fixture. There is a central core or mandrel. The cylinder is in close proximity to this—the gap being

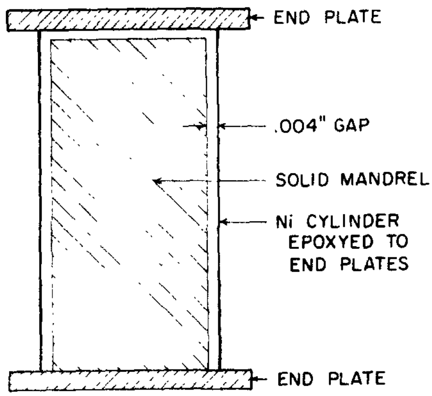

FIG. 5. Test fixture.

0·004 in. The mandrel does not carry any direct load but restricts the depth of buckle which can form. This loading fixture was mounted in a standard 60,000 lb Baldwin Hamilton test machine and load was applied through a spherical seat loading pad in the usual manner (Fig. 6).

4. RESULTS

Load was applied to the first cylindrical shell using the lowest rate of application that could be obtained with the test machine. The specimen was continuously watched and immediately it buckled the leading process was discontinued. The shell buckled in a normal diamond pattern and the buckles occurred in a small localized region. The number of buckles was counted and the value of load recorded. Next the load was increased by 50 lb. Loading was stopped and the number of buckles again counted. This procedure was repeated until the buckles had almost completely covered the surface of

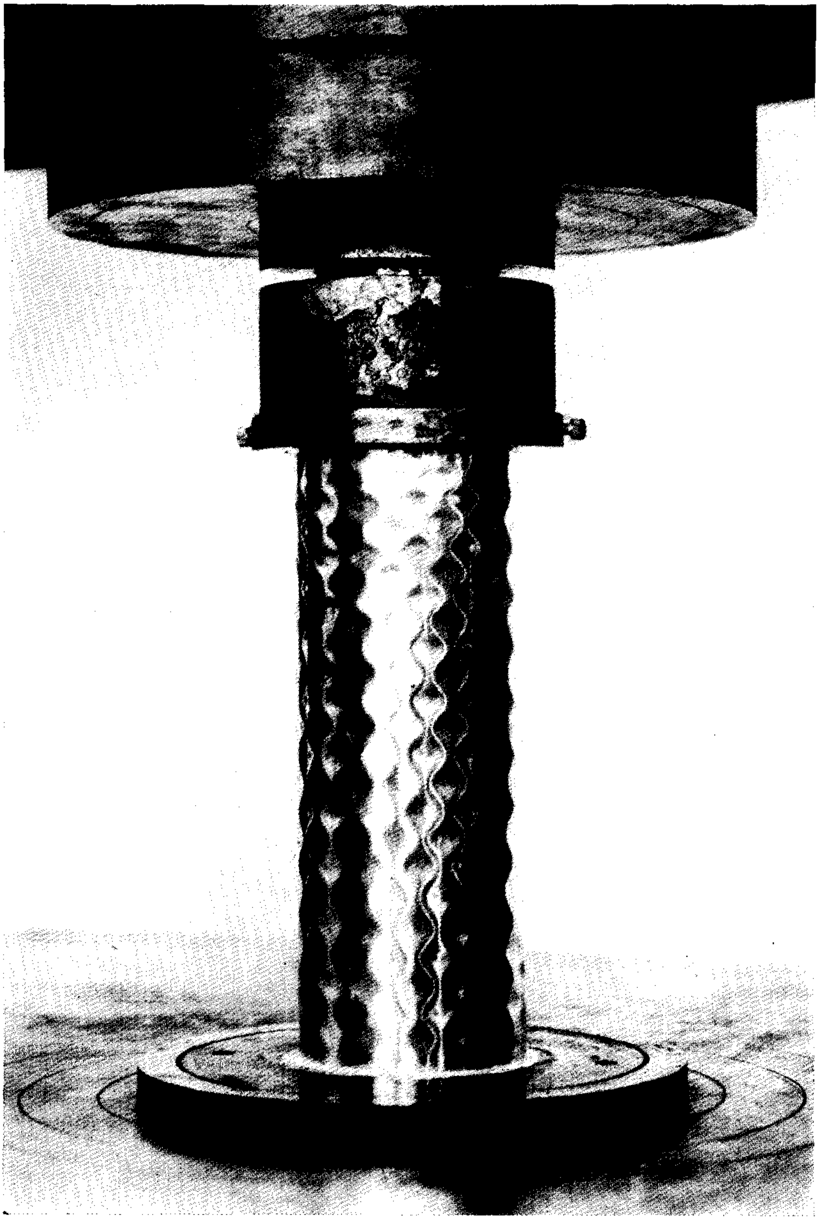

FIG. 2. Photograph of cylinder with completely developed buckle pattern.

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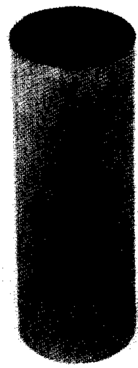

FIG. 4. Nickel cylindrical shell.

FIG. 6. Load system.

the shell. The load was now removed and the specimen was found to have returned to its initial shape. No sign of damage or permanent deformation could be found. In Table 1

TABLE 1. NUMBER OF BUCKLES VS. LOAD FOR CYLINDER NO. 1

Load (lb)	Number of buckles
995	3
1050	11
1100	21
1150	40
1200	66
1250	95
1300	161
1350	224
1400	261
1450	287

the number of buckles and the load at which the observation was made is recorded. The results are graphically portrayed in Figs. 7 and 8. These presentations show that the variation of number of buckles with load closely approximates to the Gaussian distribution, which is characteristic of a random variation. It is seen, too, very clearly from Fig. 7 that when the change in number of buckles is plotted against the load the resulting curve has zero slope at a point which is very close to the critical buckling load estimated from the classic buckling equation.

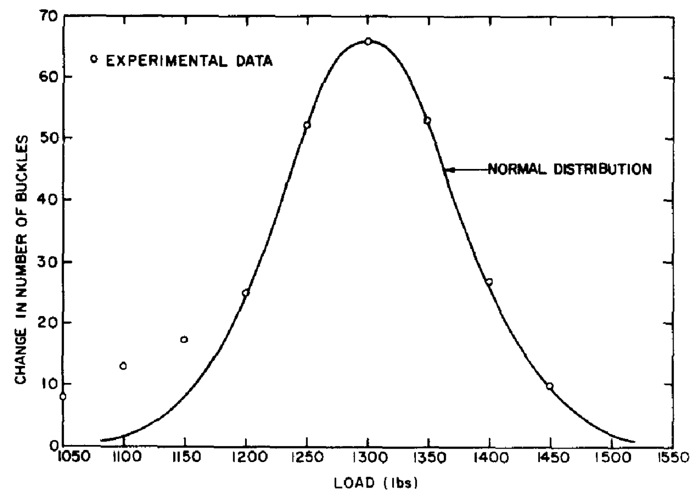

FIG. 7. Comparison of buckle distribution with normal distribution.

The extremely encouraging result obtained with the first cylindrical shell led us to believe that this method of investigation was indeed fruitful. As a consequence, a second shell was prepared in the same manner as the first. In this case the procedure adopted was as before but with addition of displacement measurements. The deflection versus load

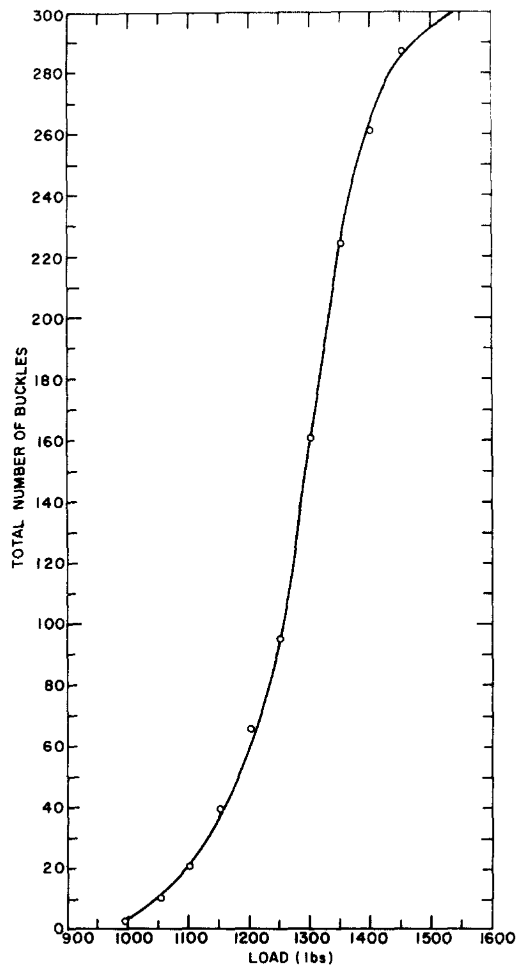

FIG. 8. Plot of total number of buckles vs. load.

history is shown in Fig. 9 and tabulated in Table 2. The buckle load behavior is as given in Table 3 and Figs. 10 and 11. Due to a slight failure of the clamping arrangements at the upper end of the cylinder the test was discontinued before the shell was completely filled with buckles. The limited data obtained, however, is fully consistent with that obtained on the previous test. There is, as one might expect however, a variation in the value of σ which defines the distribution curve. This we attribute to the fact that although the method of manufacture of the two shells was the same the second shell was of somewhat higher quality than the first.

A third nickel shell which had been prepared in the early stages of the development of the electroforming technique which was not of the same quality as the other two was also available. Despite its lower quality we thought that it would be worthwhile to test the shell and the results of this work is presented in Table 4 and Figs. 12 and 13. The shell exhibited the same behavior as had been experienced with the other two but in

TABLE 2. LOAD VS. DEFLECTION FOR CYLINDER NO. 2

Load (lb)	Deflection (10^{-4} in.)
0	0
50	3
100	7
150	11
200	15·3
250	20·5
300	25
350	28·9
400	32·2
450	36
500	39·8
550	43·2
600	47
650	50·5
700	54·4
750	58·2
800	62·2
850	66
900	70·2
950	74·1
1000	78·8
1050	83
1100	88
1150	93·6
1200	100·1
1225	103·5
1250	117
1300	129·6
1350	146·6
1400	170·8
1450	216·2

TABLE 3. NUMBER OF BUCKLES VS. LOAD FOR CYLINDER NO. 2

Load (lb)	Number of buckles
1225	2
1250	12
1300	20
1350	38
1400	104
1500	169

this case the nominal distribution curve was more platykurtic than had previously been the case. This demonstrated very forcibly to us that the kurtosis of the curve is certainly a measure of the irregularity of the cylinder.

We must emphasize that a comparison of the normal distribution curves which resulted from the three tests shows the following important facts. At the low end of the load scale there is disagreement between the experimental data and the distribution curves. This disagreement arises because the true distribution curve should result from a very long cylindrical shell whereas we tested a short one in which end effects certainly have some significance in relation to the buckle load and number at the ends of the cylinder.

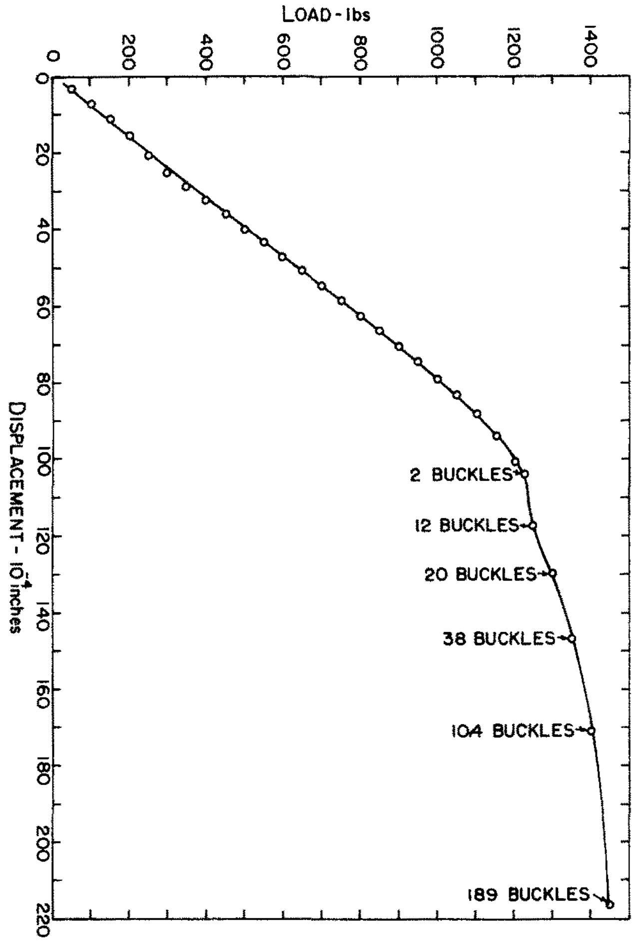

FIG. 9. Load deflection history of the second cylindrical shell.

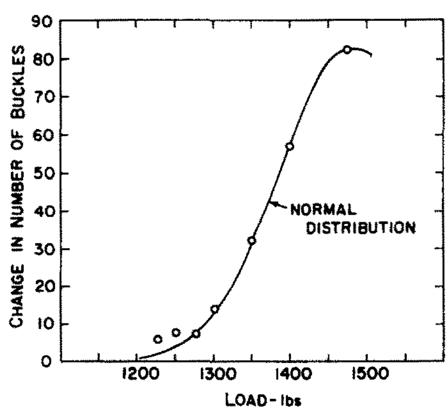

FIG. 10. Comparison of the distribution of the number of buckles with the normal distribution for the second cylindrical shell.

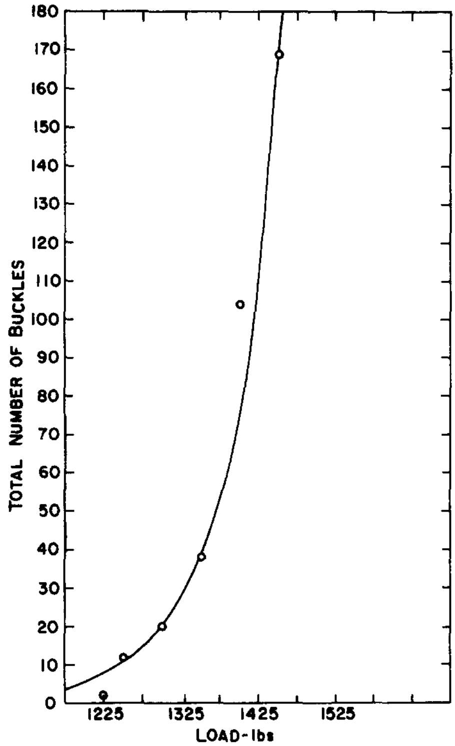

FIG. 11. Plot of total number of buckles vs. load for the second shell.

TABLE 4. NUMBER OF BUCKLES VS. LOAD FOR CYLINDER NO. 3

Load (lb)	Number of buckles
1183	13
1250	25
1300	37
1350	51
1400	92
1450	134
1500	168
1550	200

The modes for the three distribution curves are located at 1300, 1450 and 1475 lb. The computed buckling load for the cylindrical shells is 1455 lb. Thus, we see that the mode value is in no case different from the theoretical buckling load by more than 10%. A 10% variation in this value corresponds to a variation of thickness of 5% or 2×10^{-4} in.

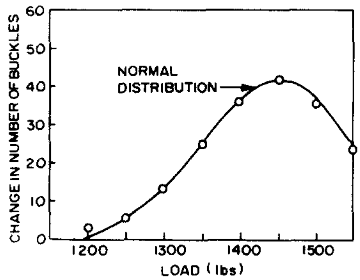

Fig. 12. Comparison of the buckle distribution number with the normal distribution for the third shell.

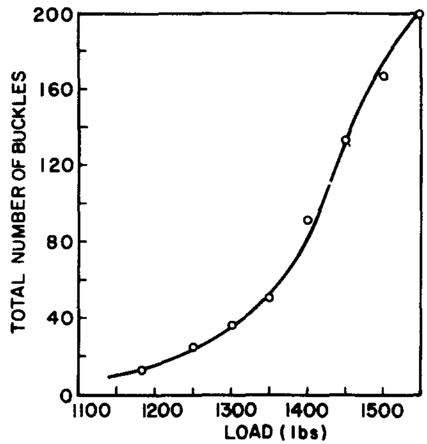

Fig. 13. Plot of total number of buckles vs. load for the third shell.

5. CONCLUSIONS

We have shown by restricting the depth to which a buckle can develop that it is possible to cause buckling to occur not only at an isolated point on the surface of the shell but over all the body. The fact that the variation in the number of buckles which occur as a function of load follows the normal distribution curve, and the mean of the smallest and greatest buckling loads approximates closely to the classical critical buckling load is clear evidence that the random distribution of imperfections which occur in cylinders is a main contributory cause to the substantial scatter which is observed in cylinder tests. This point of view is substantiated by the observations recorded in a previous paper when two distinct buckling loads were obtained from the same cylinder by the simple expedient of preventing the first buckle occurring on the second test. The closeness of the critical buckling load to the classic buckling load seems substantial support for the small displacement theory.

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APPENDIX

In this appendix we report additional work which was performed with a view to consolidating that reported in the main section of the paper.

It is variations in local geometry and mechanical property which together constitute defects. We demonstrated that such imperfections cause statistical variation in the number of buckles observed in the compressive buckling of a cylindrical shell when the buckle depth is restricted. By an argument analogous to that already developed we can show that a similar phenomena should occur in the buckling of a spherical shell or hemispherical cap when we restrict the depth of buckle. Similarly, we can deduce that a conical frustum tested under compression with buckle depth restriction should not have a variation of buckle number with load which follows the Gaussian distribution. This arises from the fact that the individual frusta which constitute the frustum tested do not have the same critical buckling load.

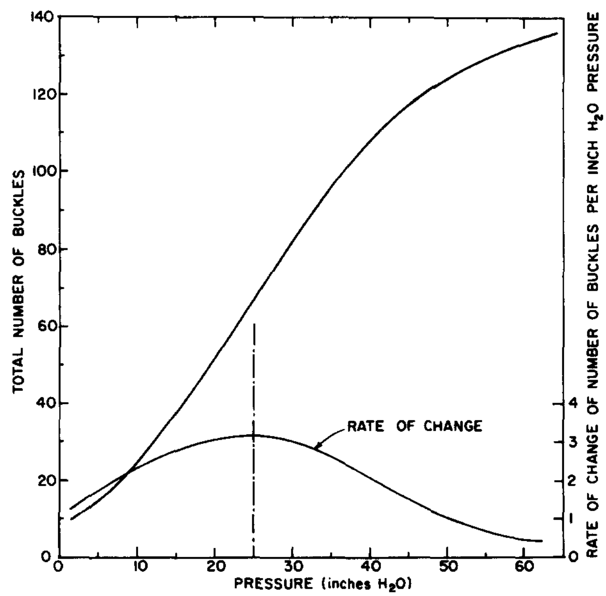

FIG. 14. Plot of cumulative and rate of change distributions of number of buckles vs. load for spherical shell.

For the first test of the new series a hemispherical shell was used. Unfortunately, this hemispherical shell had been used previously many times to demonstrate buckle patterns for such shells under external loading. Thus it without question was affected by the plastic conditions which occur at the crest of the buckles in all shells of single or double curvature which are repeatedly loaded without severe restriction on the buckle depth permitted. Thus, for this specimen the pressure to cause collapse was not in any way associatable with the critical buckle pressure as computed by any currently existing theory. On testing with buckle restriction and determining the variation of the number of buckles as a function of the external pressure it was demonstrated that the anticipated Gaussian distribution occurs in this case. The results of this experiment are given in Fig. 14.

The second test made was on a conical frustum. In this case the variation in the number of buckles as a function of load is as given in Fig. 15. It is seen from this curve

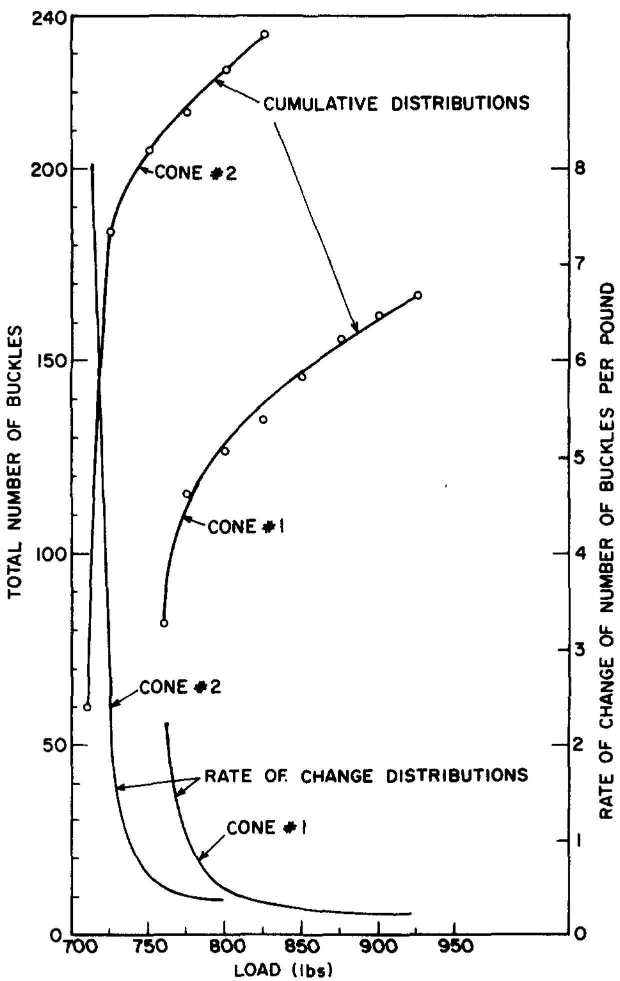

FIG. 15. Plot of cumulative and rate of change distributions of number of buckles vs. load for two conical shells.

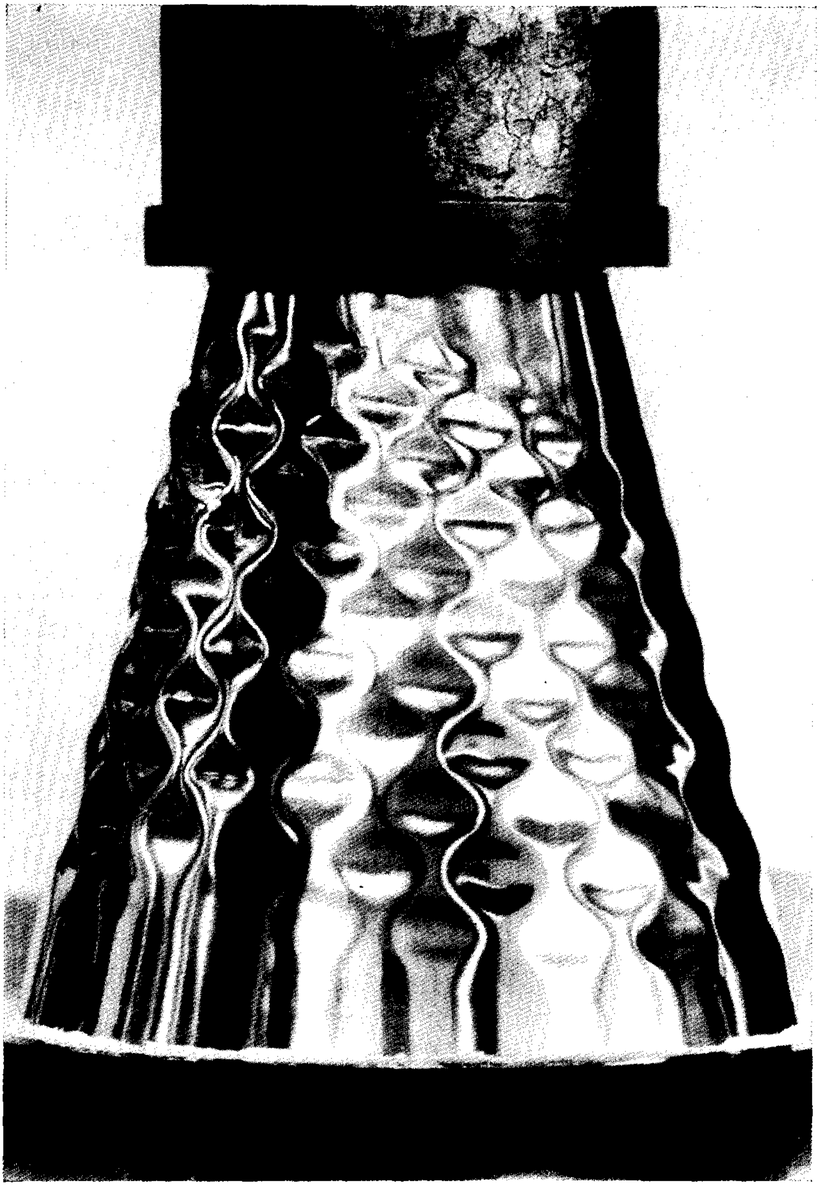

FIG. 17. Photograph of conical shell showing well-developed buckle pattern.

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that the variation is definitely not of the Gaussian type. Figure 17 shows the buckle pattern for the frustum used.

It is well known that cylindrical shells under combined loading behave differently from cylinders under pure compression. For example cylinders under compression and internal pressure buckle at a higher load than cylinders in pure compression. The limiting compressive stress however never exceeds the critical value as determined from the classic buckling formula

$$\sigma_{crit} = 0{\cdot}6\frac{Et}{R}.$$

Cylinders under compression and external pressure buckle at loads which are always lower than is achieved in pure compression. Moreover machined cylinders frequently

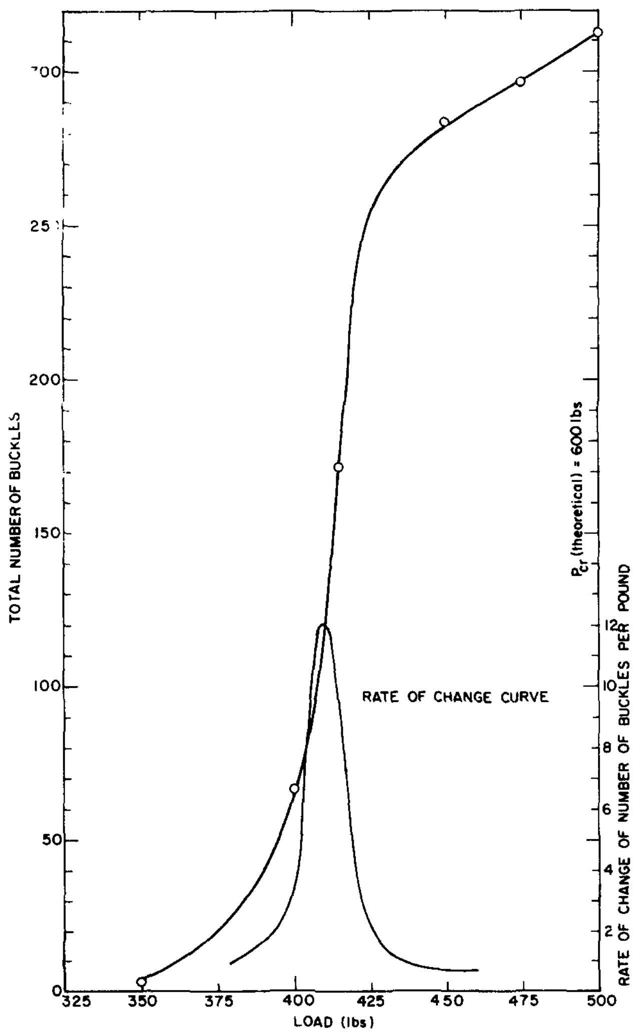

FIG. 16. Plot of cumulative and rate of change distributions of number of buckles vs. load for stressed cylindrical shell.

are prestressed and then locked-in stresses are similar to those which result from external pressure. We conjecture therefore that a normal prestressed cylinder would behave in an analogous manner to a cylinder under combined external pressure and compression, i.e. we should anticipate that such a cylindrical shell would buckle in a similar manner to a stress free shell but that there would be a variation in the position of the mode. To test this contention we constructed a thin-walled cylindrical shell by machining from a thick tube. This thin-walled shell, when removed from the mandrel on which it was fabricated was very noticeably oval in cross section. It was forced to the circular shape and tested in the same way as before, in this case we obtained a statistical variation of the number of buckles as a function of load but the mode value did not agree with the critical buckling load as determined from the classic formula (Fig. 16).

We conclude from the auxiliary tests reported in this appendix that the technique reported in the main paper is sound and the deductions therefrom are valid.

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Zusammenfassung—In diesem Bericht wird ein neues Verfahren zur Prüfung von Zylinderschalen beschrieben. Das Ausbeulen der Schale wird durch ein inwendig angebrachtes Mantelrohr begrenzt. Das Spiel zwischen dem Mantelrohr und der Zylinderschale ist gleich der Wandstärke der Schale. Durch diese Anordnung ist es möglich, die Zylinderschale zu belasten, bis sie vollkommen mit Ausbeulungen versehen ist, wobei die Ausbeulungen im elastischen Bereich bleiben. Es wird gezeigt, dass die Ableitung der Zahl der Ausbeulungen nach der Last, aufgetragen in Funktion der Last, eine Gauss'sche Glockenkurve darstellt (siehe Fig. 7). Der Maximalwert dieser Gauss'schen Kurve tritt bei einer Last auf, welche der kritischen Beullast, berechnet nach der klassischen Theorie, sehr nahe kommt.

Das Ergebnis wird als wesentliche Bestätigung der Theorie über kleine Verschiebungen betrachtet und als wichtiger Hinweis dafür, dass die Schale zuerst an der schwächsten Stelle auszubeulen beginnt und der Beulbereich von der Grösse der schwächsten Stelle abhängt.

Абстракт—В этой работе развивается новый подход к испробованию цилиндрических оболочек. Употребляется метод ограничения продольного изгиба при помощи внутренней оправки. Эта оправка распологается на расстоянии от подопытного образца равноценном толщине оболочки. Это приспособление дает возможность эластично изгибать оболочку до того что оболочка совершенно наполнена изгибами. Показано, что вариация скорости изменения числа изгибов как функция наложенной нагрузки следует распределению по Гауссу. Форма Гауссовой кривой находится на уровне нагрузки близко соответствующем критической нагрузке вычесленной по классической формуле.

Считается, что приведенные данные оказывают существенную поддержку теории малых смещений и свидетельствуют о том, что продольный изгиб имеет место прежде всего в самой слабой части оболочки и что итог изгиба является функцией площади ослабления.